On the Relation which should obtain between the Amount assured upon Lives and the Sum reserved at the Expiration of Given Terms to meet it. By CHAS. JELLICOE, *one of the Vice-Presidents of the Institute of Actuaries.*

[Read before the Institute, 27th November, 1854, and ordered by the Council to be printed.]

THE method of valuation once so universal—viz., that by means of tables of annuities involving the rates of premium charged—is now, I believe, admitted on all hands to be erroneous, and is, so far as I can learn, pretty generally abandoned. I have on more than one occasion endeavoured to demonstrate the fallaciousness of that method, and it is therefore needless again to draw attention to the peculiar consequences resulting from it; more especially as the great majority of the Companies now estimate their liabilities, as most actuaries agree that they should do, with the aid of tables based upon rates of interest and mortality approximating as closely as possible to those which observation and experience have shown to be the actually prevailing ones.

There is still, nevertheless, an important difference in the way in which this more accurate method is carried out. It is obvious that the rates of premium charged have no necessary connection with it, and that they may be altogether neglected; and in that case there can be no anticipation of the marginal addition made to the true premium for such contingencies as have to be provided for over and above the sum assured: *provided always, that the same elements are used in the valuation as those from which the mathematical or true premium was originally deduced.* For it is evident, that if in a valuation it be found necessary to depart from these elements and to adopt a lower rate of interest or higher rate of mortality, then the addition in question may be anticipated to any extent;* and if the fact of such departure be withheld, the Society so acting may take credit for the entire exclusion from its calculations of the marginal addition to its premiums, when in fact it has absorbed very nearly the whole of it. This may be done, of course, unwittingly as well as wilfully; and it is partly on account of the facility which is thus given to the production of delusive statements, and partly on account of the impropriety (as it appears to me) of

* For let the premium charged, as originally constructed, consist of $p' + \phi$, then it is evident that the adoption of either or both the alternatives mentioned will have the effect of increasing p', say, to $p' + \pi$; and ϕ is consequently diminished to $\phi - \pi$, where π may be any quantity greater or less than ϕ.

omitting all mention of so important a portion of the estimate as these additions make, that I have always advocated the valuation of them, and the introduction of their value into the account as one of its principal items. On the other hand, I am aware that some inconvenience attaches to the introduction of this value, especially in the early stages of an Assurance Company, from the circumstance of the surplus being thus rendered in appearance very large, and presenting, as is thought, a temptation to all concerned to encroach upon it. But I cannot look upon this as of any importance at the present day, when the nature of life assurance business is so much better understood than formerly; and I must still maintain, that although on this score some trifling drawback may attend the method of valuation in question, it is nevertheless by far the best that can be adopted, simply because it is the most comprehensive and at the same time the most explicit, and because it more than any other precludes all possibility of subterfuge and evasion. The objection, however, above alluded to, is so strenuously insisted upon and is so frequently reiterated, that it becomes desirable to make some effort to remove it, and to see whether the nature of the thing cannot be placed in so simple a point of view as to enable the most casual observer to form a tolerably accurate judgment himself respecting it, and to arrive at nearly correct conclusions, let the state of affairs in any given instance be presented to him in what manner it may.

It is with this object that I now propose to show the relation which should obtain between the amount assured and the sum reserved to meet it; believing that, so far as it can be depended upon, it will serve to indicate the true state of affairs as simply and directly as it is possible under all the circumstances for the thing to be done.

In every life assurance contract the person assured undertakes to pay in effect two premiums—the one to provide for the sum assured merely, and the other to create a fund for expenses and extra contingencies. The security in either case is precisely the same, and we are therefore entirely justified in applying the same principles of valuation to each. It will be convenient to keep the two distinct; and for precision's sake I will denote the value of the true premium by the expression $p'_x(1+A'_x)$, and that of the extra premium (as we may for our present purpose call it) by $\phi_x(1+A'_x)$; and I will proceed to show what portion of these must be reserved at the end of one, three, five, and seven years, in respect of assurances effected simultaneously and upon the ordinary terms,

at ages 30, 35, 40, 45, and 50, assuming that the premiums are just due at the time of each valuation, that there are no new entrants, and that each assurance continues in force throughout the term. These conditions, it is true, do not precisely obtain in practice; but I think it will be conceded in the sequel that the inferences based upon them are nevertheless fairly deduced. Taking then the "experience" for the rate of mortality, and four per cent. for the rate of interest, the following table will exhibit the portion of the extra premium to be reserved in each case, and its mean value per cent. of the sum assured at the expiration of the terms above specified.

AGE when Assurance effected.	Values of $\phi_{x+n}(1+A'_{x+n})$.			
	At end of 1 Year.	At end of 3 Years.	At end of 5 Years.	At end of 7 Years.
30	15·118	14·986	14·860	14·764
35	14·832	14·725	14·514	14·276
40	14·400	14·214	13·983	13·718
45	13·861	13·546	13·322	12·995
50	13·180	12·799	12·506	12·178
	71·391	70·270	69·185	67·931
Mean value per cent. of Sum assured ..	14·278	14·054	13·837	13·586

It will be seen by this table that the value of this portion of the reserve is never less than 12 per cent. of the sum assured, and that the least mean value is upwards of 13½ per cent. on that item. As years pass by, this mean value will no doubt diminish as respects any one class of entrants; but since in practice fresh ones are continually being added, whilst many of those first entering disappear, so that in some of the oldest Societies the average duration of the assurances has not hitherto exceeded eight or nine years, we may fairly conclude that the rate of reserve on this score, as above shown, must decline very slowly, and may often be nearly stationary for many years in succession.

What has been said applies with equal force, but inversely, to the other part of the liability, viz., that denoted by the expression $(p'_{x+n}-p'_x)(1+A'_{x+n})$, or the portion of the true premium to be set aside on account of the sum assured. Here, as will be seen by the following table, the values increase from year to year, and, as regards any one set of assurances, with great rapidity; but, for the reasons already adduced, it is probable that the mean

rate of 11·2 per cent. therein shown to prevail does not augment materially till an Office has been many years in operation. Much will of course depend on the ages at which the assurances are effected; but it will, I think, be admitted, that both this and the preceding table represent with sufficient accuracy the true state of the case in that respect.

Age when Assurance effected.	Values of $(p'_{x+n}-p'_x)(1+A'_{x+n})$.			
	At end of 1 Year.	At end of 3 Years.	At end of 5 Years.	At end of 7 Years.
30	·931	2·891	4·971	7·181
35	1·152	3·537	6·130	8·837
40	1·443	4·457	7·668	11·026
45	1·809	5·504	9·348	13·292
50	2·161	6·572	11·070	15·665
	7·496	22·961	39·187	56·051
Mean value per cent. of Sum assured ..	1·499	4·592	7·837	11·210

From what has preceded, it appears that whilst one portion of the liability of an Assurance Company is augmenting, the other has a tendency to decrease; and that, although the augmenting rate is the more rapid of the two, the consequent increase in the liability is retarded by the slow rate at which the average duration of the assurances progresses: so that there is thus reason to conclude that in many Assurance Societies the liability per cent. becomes, after a few years, nearly stationary. The following table, exhibiting the sum of the mean values given in those preceding it, will serve to illustrate this, on the supposition that the average duration of the assurances is just seven years, and that such duration has become stationary.

At end of	Values of $(p'_{x+n}+\phi_{x+n}-p'_x)(1+A'_{x+n})$.		
	Reserve to meet sum assured.	Reserve to meet extra contingencies.	Total.
1 Year.....	1·499	14·278	15·777
3 Years	4·592	14·054	18·646
5 „	7·837	13·837	21·674
7 „	11·210	13·586	24·796

By this statement it will be seen that the reserve to be made immediately on the establishment of a Company, when the value

of the premium charged is taken credit for, is very considerable, little less than 16 per cent. of the sum assured; and that this proportion increases slowly with the average duration of the policies, attaining nearly 25 per cent., or about one fourth of the total sum assured, when these last shall have become, one with another, about seven years old. The longest term hitherto reached in many of the oldest Societies is, I believe, only eight years and a half; so that the last-mentioned rate is probably not far, under ordinary circumstances, from a maximum one. It is possible, however, that assurances on young lives are more frequently dropped than those effected at more advanced ages; and if so, some allowance will have to be made on that score.

In these calculations, as I have said, the "experience" mortality has been assumed, and four per cent. as the rate of interest. If other elements be taken, the results are not so widely different as might be imagined. Thus, at three per cent. and with the "Carlisle" mortality, the table would present the following rates :—*

At end of	Values of $(p'_{x+n} + \phi_{x+n} - p'_x)(1 + A'_{x+n})$		
	Reserve to meet sum assured.	Reserve to meet extra contingencies.	Total.
1 Year.....	1·677	13·002	14·679
3 Years	5·115	12·858	17·973
5 „	8·601	12·869	21·470
7 „	12·078	13·071	25·149

A little consideration makes the seeming discrepancies in these tables sufficiently intelligible. It will be seen that the higher the rate of interest realized, the less the rate of premium required for the sum assured; and the more this last rate is reduced, the more remains for extra contingencies. The total liability in either case is often nearly the same in amount, but it is of a different quality.

What has now been said will suffice, I think, to show that there is no foundation for the somewhat prevalent notion that the proportionate liabilities of an Assurance Company, as exhibited under this aspect, are trifling at its commencement and increase only materially after the lapse of many years: I think it will be seen that such is by no means the case. The absolute liability, of course, augments rapidly as the transactions entered into are

* These are the mean values in respect of assurances effected at each quinquennial interval from 25 to 55 inclusive.

multiplied; but as regards each one hundred pounds assured, it plain that where credit is taken for the value of the full premium, at least fifteen pounds should be reserved at the very commencement, and that this proportion must gradually increase till it becomes nearly double—when, if we may judge by our past experience, there is reason to think that a pause takes place, and that the proportion in question approximates to its maximum value. This, however, I need scarcely repeat, must depend, in any given instance, entirely on the facts of the case.

Of course what is here said applies exclusively to that system of valuation which regards the total liability, and which takes into account the value of the extra as well as that of the true premium. Where the former is disregarded, and the latter alone included in the estimate, the reserve should evidently be in correspondence with the per centages exhibited in the first of the three columns of the table at page 103, or of that at page 104. It is in this shape that the results of a valuation by the Northampton Table and other similarly constructed ones have been usually made to appear; and it may be remarked, that the results so obtained are true only when the rate of mortality and rate of interest actually prevailing correspond with those of the adopted table—in which case it is clear that the whole of the premium charged is absorbed, and that there is no provision for extra contingencies at all.

From these considerations, then, it appears that the reserve made by Assurance Societies may be and is exhibited in several ways; and since it is of some importance to make the due distinction between them, I will briefly point out some of them, and, for the sake of greater perspicuity, give them a general expression. We may thus represent the first by

Dr.	Cr.
$p_{x+n}(1+\mathrm{A}_{x+n})$	$p_x(1+\mathrm{A}_{x+n})$,

where p_x represents the premium charged, and A_x the annuity corresponding with it, and where the results would consequently be those obtained by the use of tables formed as the Northampton and some few others are. For the second mode we may take the formula

Dr.	Cr.
$p'_{x+n}(1+\mathrm{A}'_{x+n})$	$p'_x(1+\mathrm{A}'_{x+n})$

in which the true premium alone is involved, and the additions altogether disregarded. Here the rates of reserve would correspond with those shown in the first column of per centages, at pages 103 and 104. For the third may be selected the expression

Dr.	Cr.
$(p' \pm \pi)_{x+n}(1+\mathsf{A}'_{x+n})$	$(p' \pm \pi)_x(1+\mathsf{A}')_{x+n}$

which will serve to indicate the results when the premium originally adopted as the true one has been departed from, and the value of the additions enhanced or encroached upon, whether disregarded or otherwise, accordingly as the sign + or — is really applicable. A fourth method may be exhibited by the formula

Dr.	Cr.
$(p'+\phi)_{x+n}(1+\mathrm{A}'_{x+n})$	$p'_x(1+\mathrm{A}'_{x+n})$

The rate of reserve will here be similar to that in the last column of the tables at pages 103 and 104, and will comprise the value of the future additions as well as that of the liability under the sums assured. Lastly, a mode adopted by some few Societies whose manner of distributing the surplus is peculiar to them may be denoted by the expressions

Dr.	Cr.
$p'_{x+n}(1+\mathrm{A}'_{x+n})$	$(p'+\phi)_x(1+\mathrm{A}'_{x+n})$

where, as will be observed, the true value of the whole premium charged is set against the true value of the sums assured, and where, in consequence, the amount reserved appears to be out of all proportion to what it should be. It is proper however to mention, that in the Societies adopting this arrangement the value of the $\phi_x(1+\mathrm{A}'_{x+n})$ is, in fact, reserved, and forms the fund out of which the reductions in the future premiums are provided for: that is to say, the additions originally made to the true premium being found to be worth so much, and that sum being forthcoming over and above what is required for other purposes, it would be argued that such additions might be dispensed with, and the true premium alone accepted on account of the future payments to be made.

If a Society could be conducted without expenses of any kind, it would be reasonable to expect *à priori* that the true premiums alone would suffice; or, in other words, that the reductions allowed would be exactly equal to the additions originally made. As however expenses must be incurred, and, under ordinary circumstances, to a considerable amount, such a result is hardly to be looked for —unless, indeed, the funds of the Society are augmented from other sources than the usual ones, or unless the premium assumed in the outset to be the true one be ascertained by subsequent investigations to be more than sufficient for the purpose, and a reduced rate be substituted in after valuations.